POLITECNICO DI MILANO

    

Introduction to Decision Models

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To decide …

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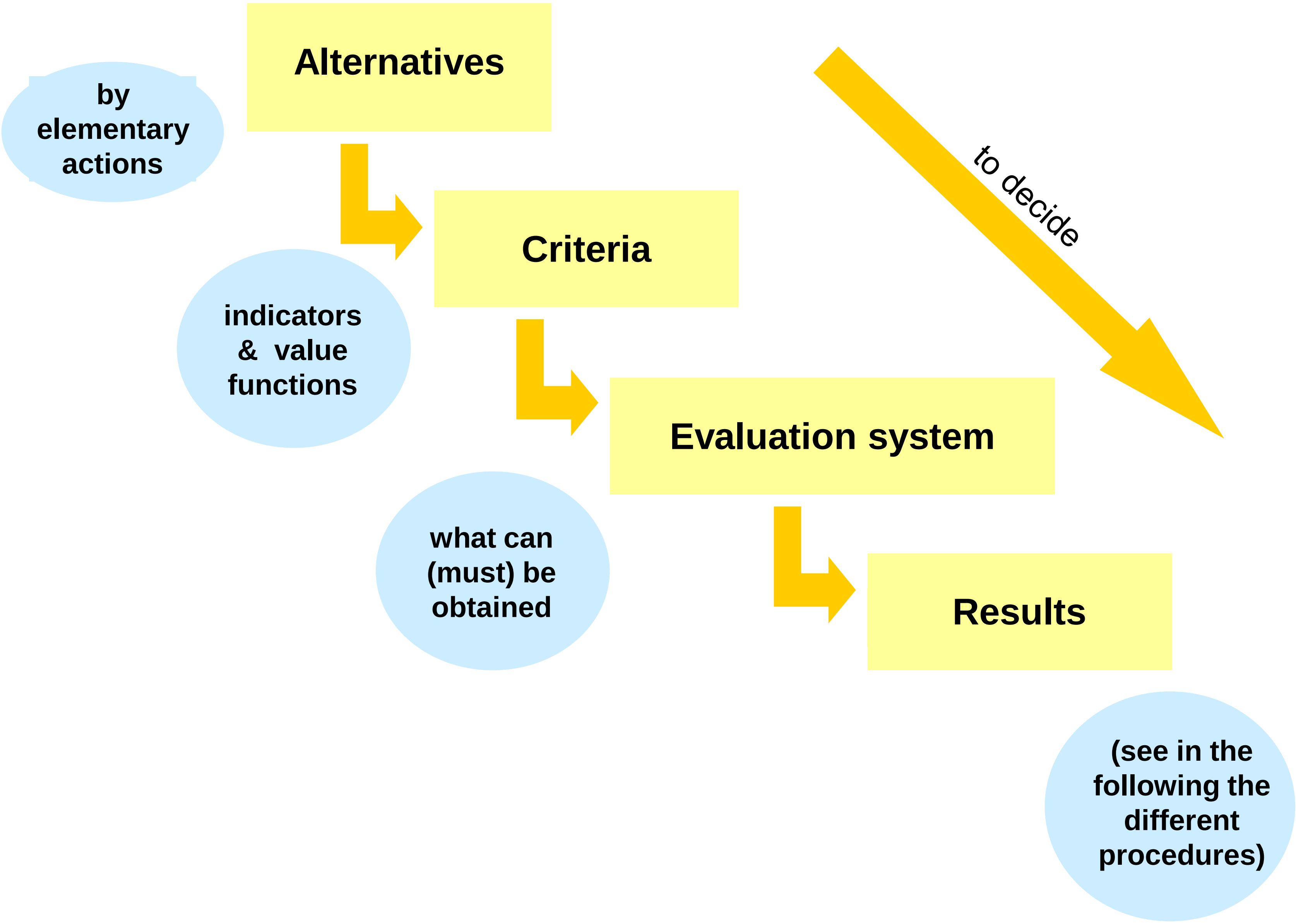
The steps of a decision



The different levels of a decision process

You want to go outside to dinner with your wife, so …

i. **Information →**	Let's go out for dinner.
ii. **Feedback →**	Let's go out for dinner, do you agree ?
iii. **Discussion →**	Let's go out for dinner, where can we go ?
iv. **Involvment →**	Would you like to go out ? to do what ?

- different actors (Decision Makers, DM's)
- a (possibly pre-defined) procedure
- constraints *(is the restaurant open ?)*

Decision Theories: a brief introduction

Short history:

- 40's → Genesis (during the 2° war)
- 50-60's → **Development [*]** (LP probl. & Combinatorics)
- 60-70's → Specialization (non linear, integer, B&B, …)
- 70-80's → Multicriteria (the importance of trade-off)
- 50-90's → Multiple DM (the different points of view)
- 80-00's → Decision Aiding (sw supporting the process)

[*] max f(x), s.t. x Є X (X finite or infinite set)

Links & references:

- http://www.informs.org (the INFORMS site)
- http://www.euro-online.org (the EURO site)
- http://www.airo.org (the AIRO site)
- http://www.journals.elsevier.com/european-journal-of-operational-research/ (EJOR, a major OR journal)
- Tsoukias A., *From decision theory to decision aiding meth.* (EJOR, 2007)

An "ideal" decision problem

- **Someone who decides**

  with respect to one clear **objective**

  with a set of well defined **constraints**

  with all the suitable **information**

  in presence of a finite / infinite **set of alternatives**

- **Examples**

Ideal example 1

Combinatorial optimization

Your chorus is defining the storyboard of a concert and you must choose between a set of mottetti (a “mottetto” is a choral musical composition).

Each mottetto (m_1, m_2, …, m_n) has a time of execution t_j and a level of success s_j ($j = 1,\dots,n$).

The total time of the exhibition is T min.

What can you do ?

If you want, consider this specific instance:

$n = 4$; $t = (10, 22, 37, 9)$; $s = (60, 55, 100, 15)$; $T = 45$

(i) What are the variables ?

(ii) How many solutions ?

(iii) What is the optimal choice ?

Ideal example 2

Linear programming (LP)

You must define the week production of a (small) firm that has only 2 products, A and B.

One item of A needs 4 units of the resource R1 and 2 unit of the resource R2.

One item of B needs 1 unit of the resource R1 and 3 units of the resource R2.

You have (weekly) 200 units of R1 and 480 units of R2, and you know that the maximum possible sale for B is 110 items.

The net revenue for item A is 500 €, for item B is 300 €.

What can you do ?

(i) What are the variables ?

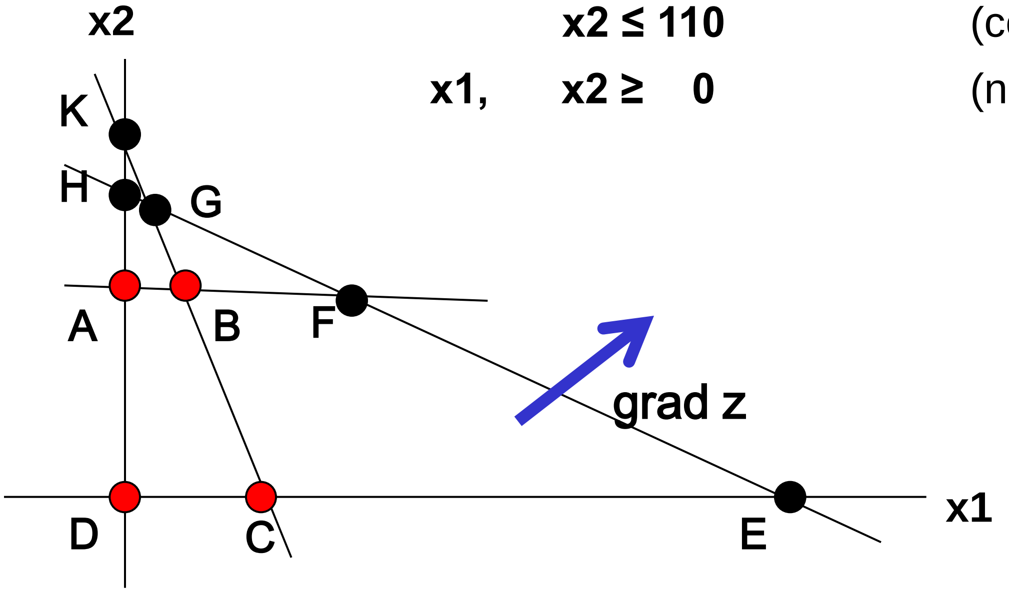
(ii) How many solutions ?

(iii) What is the optimal choice ? *(you can solve with Excel)*

Ideal example 2: the model

LP properties …

z (max) = 500 x1 + 300 x2	(objective function)
s.t.	(constraints)
4 x1 + 1 x2 ≤ 200	(resource R1)
2 x1 + 3 x2 ≤ 480	(resource R2)
x2 ≤ 110	(constr. x2 sale)
x1, x2 ≥ 0	(non neg. constr.)



What is the optimal choice ? (http://gim.altervista.org/ro/)

A real decision problem

- **Uncertainties** (non-deterministic context, data mining)
- **Complexity** (problem dimension, non linearity, …)
- **Several stakeholders** (distributed decision power)
- **Different rationalities** (criteria and preferences)
- **Various time horizons** (often)
- **Use of simulation models**

  ↳ what … if …

A frame for decision problems

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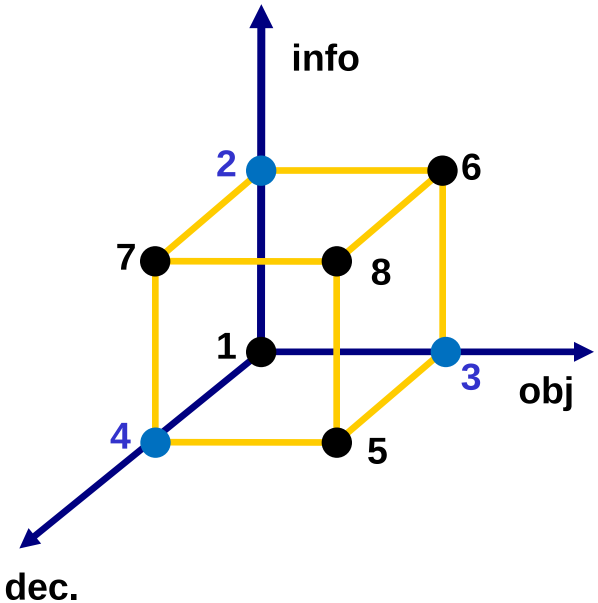
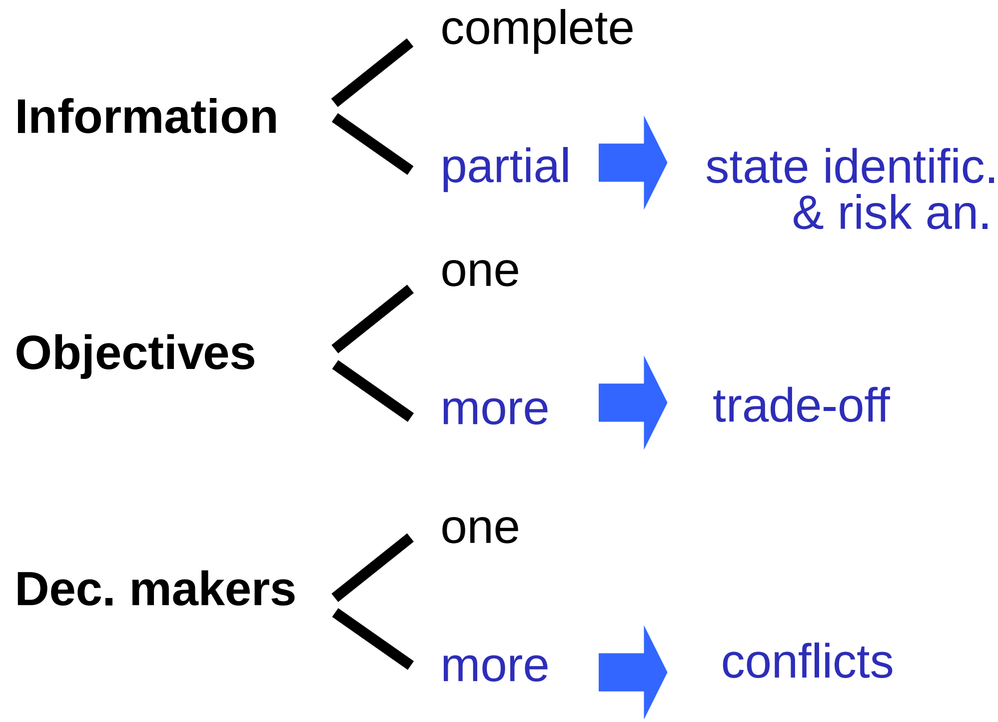
Decision processes: the (3) main elements





1. Math. programming
2. **Risk analysis**
3. **Multiple criteria**
4. **Social choice**

5, 6, 7, 8 → Game theory, …

A real decision process

- **Uncertainties** (non deterministic context, …)
- Complexity (problem dimension, non linearity, …)
- Several stakeholders (distributed decision power)
- Different criteria (preferences)
- Different time horizons (often)
- Use of simulation models

  what … if …

The **structure of preferences** of the decision maker(s)

- The **perception of the problem** and the differences between

  normative approach

  cognitive approach

The perception of the problem

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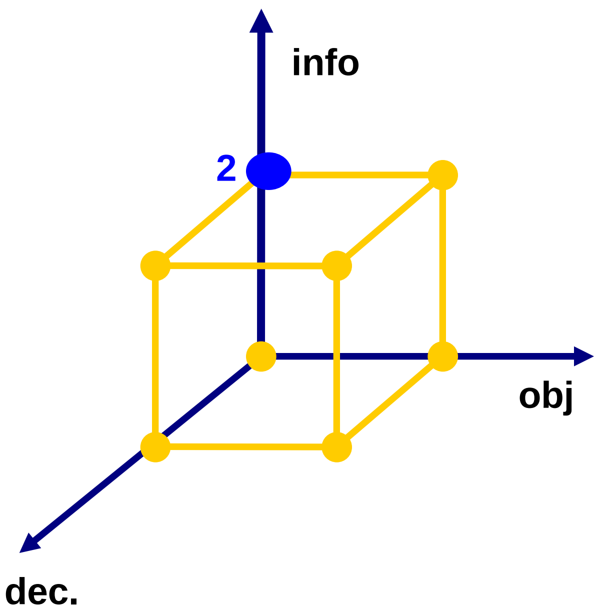
Decision process in a non-deterministic context



Information
- complete
- partial **[*]**

Objectives
- one
- more

Dec. makers
- one
- more

1. Math. programming
2. Risk analysis
3. Multi-objective (criteria)
4. Social choice

5, 6, 7, 8 → ….

[*] → non-deterministic context

perception & mental models

Two (opposite) theories

(a) Normative theory (prescriptive) → what the DM **should do**

(b) Cognitive theory (descriptive) → what the DM **really does** → experimental tests

When they are the same ? ➡ if the (***single***) DM has all the information (***in a deterministic way***) and has clearly in mind *the* criterion *(**one**)* of evaluation

⬇

optimization (easy)

N-1° **Principle of INVARIANCE**

Equivalent (from the logical point of view) versions of the same problem **must** produce the same choice

Examples

- Change names or positions for the options
- Change measure units
- Add a constant value for all the results

Counterexamples

- Lotteries (cases A, B, C)
- Ellsberg paradox (1961)

N-2° Principle of DOMINANCE

If the DM prefers A with respect to B in every scenario (or context or state of nature) the choice **must be** A

Examples

- I prefer to be missionaire (with respect to engineer) in peace and prefer to be missionaire (...) in war
- I prefer chicken with respect to beef (when there is nothing else) and I prefer chicken ... also when there is fish

so the choice ... is better then the choice ...

Counterexamples (see in next lessons)

- Extraction from an urn filled with 100 balls (Tversky, Kahneman, 1986)
- The possible choices in uncertainty conditions (with the DM risk attitude)

Extractions …

n. of balls	situation A	situation B
90 white	0	0
6 red	45	45
1 green	30	45
1 blue	-15	-10
2 yellow	-15	-15

n. of balls	situat. C	situat. D	n. of balls
90 white	0	0	90 white
6 red	45	45	7 red
1 green	30	-10	1 green
3 yellow	-15	-15	2 yellow

Better A or B ?

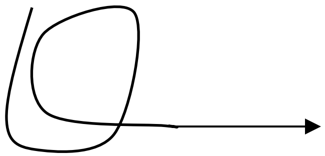
better …

Better C or D ?

better …

Cognitive theory: a first principle

C-1° **Principle of NON NEUTRALITY**



The aggregation of (decisional) options **is not** a neutral operation !

Given the two preferences on A1 and B2, it is **not guaranteed** that their aggregation (C1) is the preferred one

- **Caution: do not combine too easily the options**
- **Normally, the ambiguity is avoided, “even if this is not rational " (Ellsberg)**

Cognitive theory: three more principles

C-2°

Principle of EVIDENCE

The dominance among options should be **obvious**

C-3°

Principle of ASYMMETRY

Possibility of **losing** K is more important than **winning** K

C-4°

Principle of COMPACTNESS

An aggregated option (A) has an importance less than the sum of the importances of the single sub-options (A_1.A_2)

$$\pi(A) < \pi(A_1) + \pi(A_2)$$

The structure of preferences

(multiple criteria)

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Example – A sabbatical year

- Professor AC has to decide where going for a sabbatical year (he has 5 options)

- Data are the following:

a discrete case

	Rome	Berlin	Geneva	Moscow	Tokio
Reward (k€)	5	7	10	2	7
Univ. prestige	3	9	4	6	5
Life quality	10	4	5	3	3

- Qual. (or quant.) scales, converted in numerical [0, 10] ones (**Ph.1**)
- Search for the best choice, between the 5 alternatives (**Ph.2-3**)
- A multi-criteria (discrete set of options) decision problem

Dominance

	Rome	Berlin	Geneva	Moscow	Tokio
Reward	5	7	10	2	7
Univ. prestige	3	9	4	6	5
Life quality	10	4	5	3	3

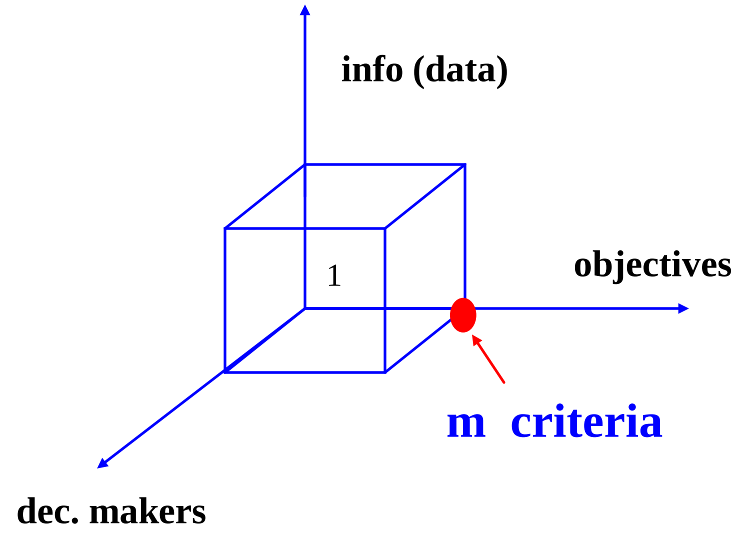
a common scale !

- Comparison bertween Berlin and Tokio
- Berlin dominates Tokio → why ?
- Definition (1) *dominance* → in a dec. problem with m objectives (criteria) to be maximized, max $c_1(x)$, …, max $c_m(x)$, a solution **x** dominates a solution **y** if $c_1(x) \geq c_1(y)$, …, $c_m(x) \geq c_m(y)$, that is the solution **x** obtains better (or equivalent) results with respect to the solution **y** for all the objectives.
- Definition (2) *efficient solution* → a solution **x** *non dominated* by any other solution is called *efficient* or *Pareto* solution.

More about dominance

- In this context it is still valid the concept of dominance ? YES
- There are → 2 dominated solutions (why ?)
  3 efficient (non dominated) solutions
- If the data are correct & if the teacher is rational, he must choose only between Rome – Berlin – Geneva (non dominated sol.)
- So AC has reduced the options, but he doesn't already chosen the final solution (Ph.2 is done, but Ph.3 no → we need …)
- What option ? It depends on the *importance* that the teacher acknowledges to the various criteria: economics (*Reward*), working place (*Univ. prestige*), environment (*Life quality*)
- The preference structure of the DM could be very complex; but in the simpler case it is a vector with dimension equal to the number of criteria (3 in this case)

The reference frame



- Three axis
- The 1/m/d case → Decision with m objectives (m criteria)
- Formulation → Min or max with $x \in X$ → $\begin{vmatrix} c_1(x) \\ c_2(x) \\ \vdots \\ c_m(x) \end{vmatrix}$

  (a vector of obj. functions)
- Problems
  - **continuous case → multi-objective analys**
  - **discrete case → multi-criteria analys**

Three phases of the choice (more in details)

- **Phase 1 → Data analysis**
  - the objectives of the decision maker are measured by functions
  - each function shows the value of an indicator
  - each indicator has his own unit of measure
  - to compare them ***a common scale*** is needed
  - the scale exhibits the ***utilities perceived*** by the decision maker

- **Phase 2 → Efficient solutions**
  - are there some dominated solutions ?
  - ***elimination*** of the dominated solutions
  - not dominated or Pareto or ***efficient solutions*** (synonyms) remain

- **Phase 3 → Final choice**
  - analysis of the ***preference structure*** of the decision maker
  - vector of weights (pair comparison)
  - weighted sum of the utility of each alternative
  - ranking, final choice, ***sensitivity***

Phase 1 – Indicators (and their units of measure)

- **Example of the incinerator :**

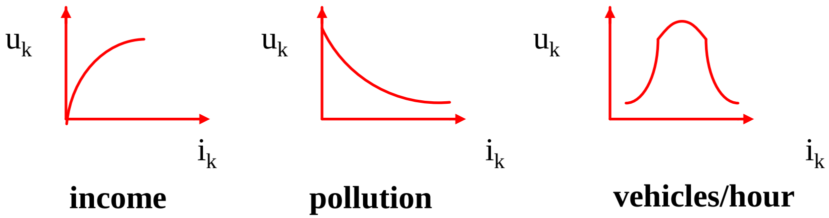
  max f_1 (profit) → in millions €/year

  max f_2 (air quality) → fraction between 2 values in mg/m^3

- **What:** to analyze the link between a certain indicator and ***utility perceived*** by the decision maker → a function $u_k(i_k)$, where i_k represent the value of the indicator related to the objective-function $f_k(x)$

- **Why:** the **utility function** u_k allows to affirm that the solution $\hat{x}$ is better than the solution $\bar{x}$ (following that objective-criterion) if $u_k(\hat{x}) > u_k(\bar{x})$, while there is no preference if $u_k(\hat{x}) = u_k(\bar{x})$

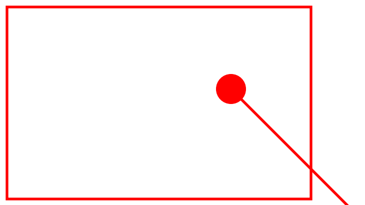
- **Examples of utility functions**



Phase 2 – Evaluation matrix

- **Discrete case:** **Multi Criteria (MC) Analysis**
  - a finite number (n, usually small) of alternatives
  - a finite number (m) of criteria

- **Evaluation matrix**



rows (m) → criteria

columns (n) → alternatives

u_{kj} = utility with respect to criterion k of the alternative j

- **Example (sabbatical):**

$$\begin{array}{l} \\ \text{Reward} \\ \text{Univ. prestige} \\ \text{Life quality} \end{array} \quad \begin{array}{c} \begin{array}{ccccc} R & B & G & M & T \end{array} \\ \begin{pmatrix} 5 & 7 & 10 & 2 & 7 \\ 3 & 9 & 4 & 6 & 5 \\ 10 & 4 & 5 & 3 & 3 \end{pmatrix} \end{array}$$

Values in a common (conventional) scale

Phase 3 – The final choice

- One more element → → → → the preferences structure (weights)

- Matrix

(*) dominated

	Rome	Berlin	Genève	Moscow (*)	Tokyo (*)
Reward	5	7	10	2	7
Univ. prestige	3	9	4	6	5
Life quality	10	4	5	3	3

Evaluation matrix

weights
0.3
0.6
0.1

weights

- The vector of the weights measures the importance that the DM gives to the criteria

- **Weighted sum**

R	B	G	M	T
4.3	**7.9**	**5.9**	**4.5**	**5.4**
5°	**1°**	**2°**	**4°**	**3°**

These values (the total utilities) are acalculated as a sum of products: **rows x weights**

- What does it mean ? What is his use?
  - ↳ satisfaction related to each alternative
  - ↳ ranking the alternatives, so giving the choice → → → **Berlin**

Phase 3 – Subjectivity (the wife influence or …)

- A factor of influence for the DM → his wife
- Change the structure of preferences

  ↳ the wifes gives much more importance to the life quality (and much less importance to the university prestige)

0.4
0.1
0.5

- Wife weighted sum and new ranking

R	B	G	M	T
7.3	5.7	6.9	2.9	4.8
1°	3°	2°	5°	4°

the wife influence pushes for **Rome**

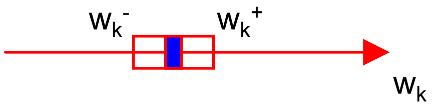
- Conclusion: though the use of the same data (eval. matrix) different preferences can make different choice → it depends on the weights

subjectivity

but note that a dominated alternative cannot win (for any weight set)

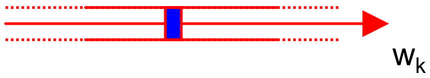
Sensitivity and RR (Rank Reversal)

- **Goal:**
  - To find the variations w_k^+ (increasing) e w_k^- (decreas.) of the weight of the k^{th} criteria w_k within which the choice doesn't change (the alternative in the first position remains)
- **Method:**
  - keep all the weights w_i ($i=1,...,m$; $i \neq k$) except w_k with the values given by the DM and calculate the overall utilities of the alternatives as functions of w_k
  - calculate the values of w_k given which the alternative ranked first keep having the higher utility
- **Result:**
  - "narrow" range, little changes in the weight w_k would cause a different choice of the alternative



Narrow range

  - "wide" range, big changes in the weight w_k **wouldn't** cause a different choice of the alternative



Wide range

Example

A multicriteria decision problem (6 alternatives, 3 criteria = utilities) is showed in this matrix, with its weight vector.

	a_1	a_2	a_3	a_4	a_5	a_6
c_1	60	40	20	70	100	80
c_2	40	40	35	35	35	40
c_3	20	30	60	40	50	50

w_1	0.20
w_2	0.40
w_3	0.40

1. Are there dominated alternatives ?
2. What is the ranking and the final choice ?
3. Is the result changing if w_2 increase ? Is there a Rank Reversal ?

Tools

A formal decision process needs instruments for:

i. **abstraction**

ii. **analysis**

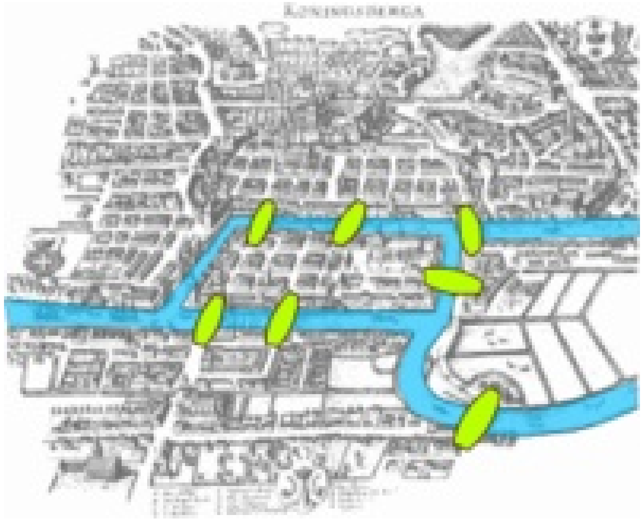
iii. **synthesis**

(and more …)

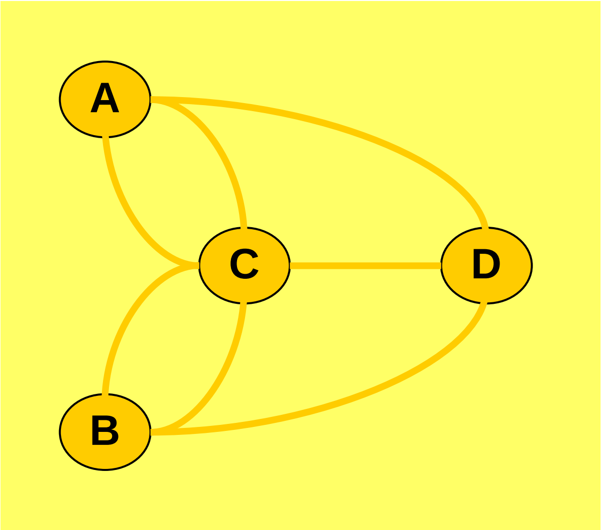
Tools for abstraction

- 1736
- Konigsberg

- Euler
- **Graph theory**



The problem



The model

Graph theory & decision problems

- **General reports**
  - http://en.wikipedia.org/wiki/Graph_theory
  - http://en.wikipedia.org/wiki/Route_inspection_problem
  - http://teoriadeigrafi.altervista.org/teoria_dei_grafi.pdf (a tutorial)
- **Applications**
  - http://www. …
  - http://www. …
  - http://www.ratp.fr/plan-interactif/ (the Paris metro)

search …

- **A famous problem – TSP**
  - http://www-e.uni-magdeburg.de/mertens/TSP/index.html
  - http://www.tsp.gatech.edu/index.html
  - http://www.graphtheory.com/

Tools for analysis

- Sudoku (Corriere della Sera, 3 Sept. 2010)

		4				9		
	1	6	2		4	3	8	
	8						5	
4			6		2			1
3			9		8			4
	3						6	
	6	7	3		5	1	4	
		2				8		

- Rules …

- **Branching** (a lot of small subproblems)

Tools for analysis …

Step 2

		4				9		
	1	6	2		4	3	8	
	8					4	5	
4			6		2			1
3			9		8			4
	3						6	
	6	7	3		5	1	4	
	4	2				8		

Step 4

		4				9	1	
	1	6	2		4	3	8	
	8	3				4	5	
4			6		2			1
3			9		8			4
	3						6	
	6	7	3		5	1	4	
	4	2				8		

Step 6

		4				9	1	
	1	6	2		4	3	8	7
	8	3				4	5	
4			6		2			1
3			9		8			4
	3						6	
	6	7	3		5	1	4	X
	4	2				8		

What number in position X ? → 2 or 9

branch (a) → X = 2

but if X = 2, there is no place for a 2 in the right-high block;
so X = 2 → NO

branch (b) → X = 9

in this case …

Tools for analysis …

Step 8

		4				9	1	
	1	6	2		4	3	8	7
	8	3				4	5	
4			6		2			1
3			9		8			4
	3						6	
8	6	7	3	2	5	1	4	9
	4	2				8		

Step 9

		4				9	1	
	1	6	2	**Y**	4	3	8	7
	8	3				4	5	
4			6		2			1
3			9		8			4
	3						6	
8	6	7	3	2	5	1	4	9
	4	2				8		

What in the position Y ? → *5 or 9*

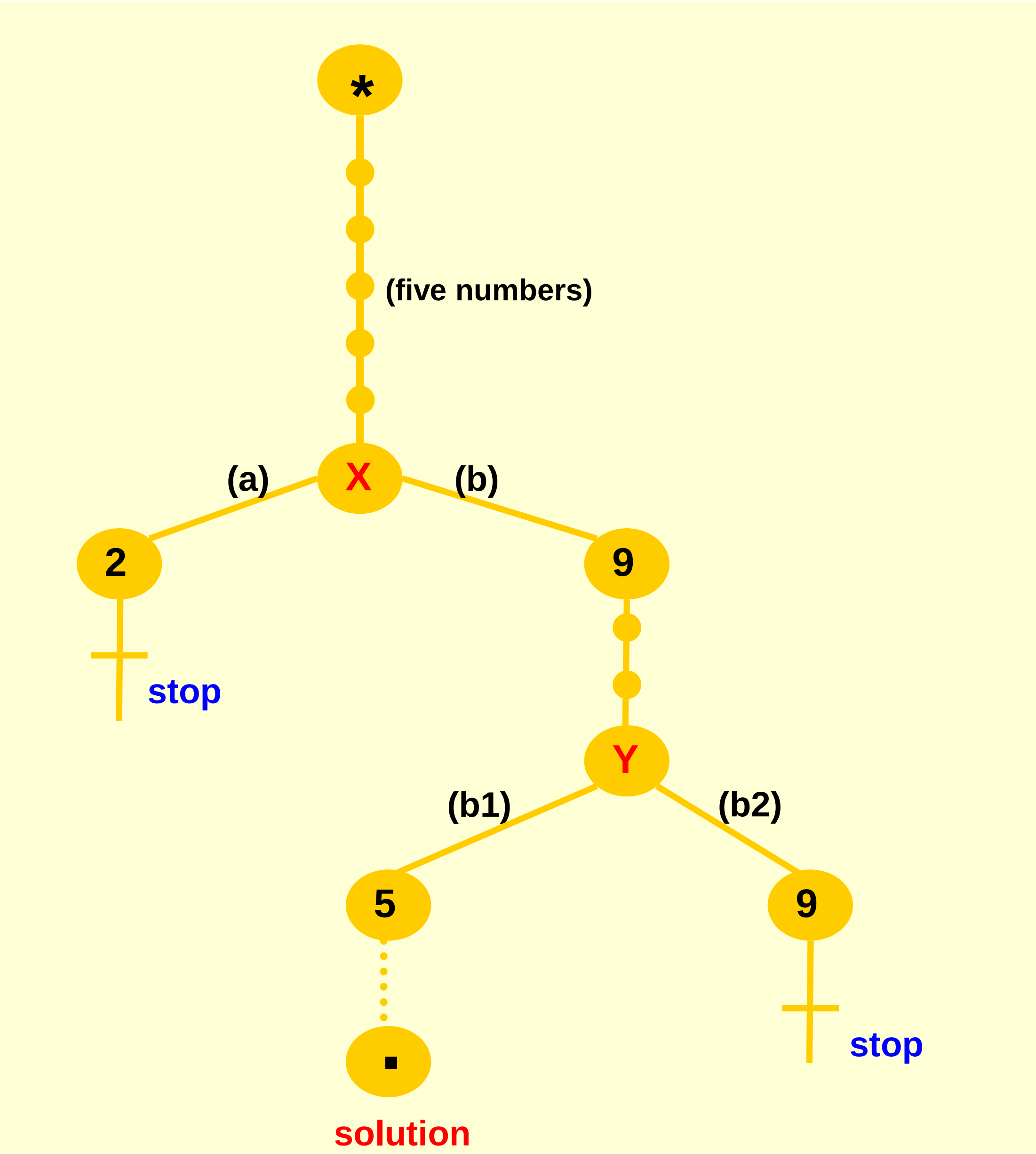
branch (b1) → ***Y = 5***

in this case …

branch (b1) → ***Y = 9***

in this case …

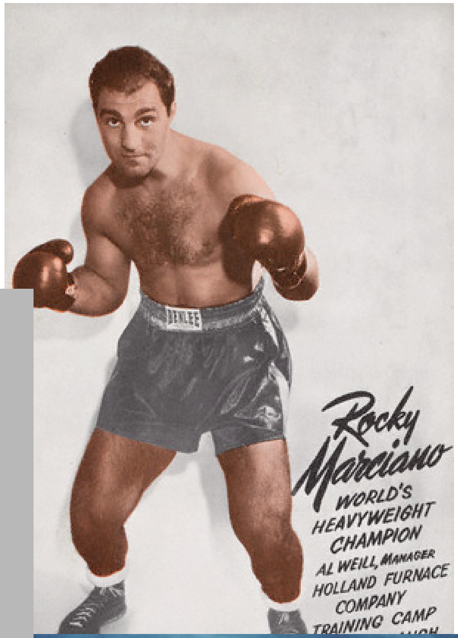
The solution (visualization)

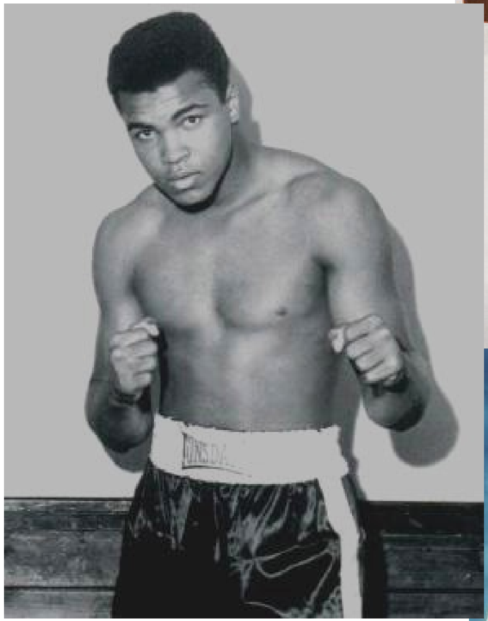


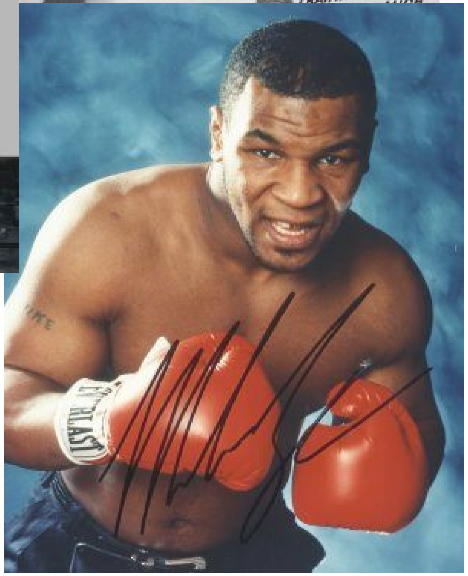
- Branching rules (X=…)
- A lot of (easier) subproblems
- Stopping rules (no sol.)

Tools for synthesis

Who is the all time world's best boxeur ?







Indicators:

- strength
- speed
- n. of victories
- years of premiership
- …

We need a common framework to compare the alternatives !

Contact

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Consorzio Poliedra

- www.poliedra.polimi.it

Thank you for tour attention